MATH 220.204, APR 1 2019

1. Let ζ denote the Riemann zeta function, and suppose that $\zeta(z) = 0$ for some $z \in \mathbb{C}$. Prove that $\mathrm{Re}(z) = 1/2$.

2. Prove that every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

3. Let X be a nonsingular complex projective manifold. Then every Hodge class $\alpha \in \mathrm{Hdg}^k(X) = H^{2k}(X;\mathbb{Q}) \cap H^{k,k}(X)$ is a rational linear combination of algebraic cocycles.

4. Prove that for any compact simple gauge group G, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has mass gap $\Delta > 0$.

5. Let $\mathcal{E}$ be an elliptic curve over a number field K, and let $L(E, s)$ be the associated L-function. Prove that the rank of $E(K)$ is the order of the zero of $L(E, s)$ at $s = 1$.

APRIL FOOL'S!

1. Consider the sequence defined by:
$$\begin{cases} u_0 = \dfrac{1}{2} \\ u_{n+1} = \dfrac{u_n + 1}{u_n + 2} \qquad \text{for } n \geq 0. \end{cases}$$
Prove that $0 < u_n < \frac{2}{3}$ for every integer $n \geq 0$.

2. Let n be a positive integer, and let $\mathbb{Z}_n$ be the set of integers modulo n. Let
$$S = \{[x] \in \mathbb{Z}_n : [x^2] = [x]\}$$
   (a) Write out the elements of S when $n = 15$. You may write this as a list $\{[a], [b], [c], \ldots\}$.

   (b) Prove that if n is prime, then $S = \{[0], [1]\}$.

3. Let F_n be the Fibonacci sequence:
$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{if } n > 2.$$
Prove that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$.

4. Suppose that $f : A \to B$ is a function and $C \subseteq B$. Prove that $f(f^{-1}(C)) = C \cap f(A)$.

5. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by
$$\forall n \in \mathbb{N}, \quad f(2n-1) = 3n - 2 \qquad f(2n) = 3n - 1$$
Prove that $\mathbb{N} \times \mathbb{N}$ is denumerable by showing that the function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined as $g(m, n) = 3^{m-1} f(n)$ is bijective.